Adaptation to the edge of chaos in one-dimensional chaotic maps

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We present a method that enables chaotic systems to change its dynamics to stable periodic dynamics by a feedback adjustment. The proposed method uses feedback of a largest value obtained from observations of a fixed interval of time series of the system variable and therefore does not require any *a priori* detailed information. We apply this method to several chaotic systems and confirm numerically that chaotic states are stabilized to stable periodic ones. Since the stabilized states in the system are formed around a boundary between regular states and chaotic ones, the method provides a kind of adaptation to the edge of chaos.

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During the last few decades, lots of studies on chaos control have been conducted despite many intractable properties of chaos, e.g., unpredictability, sensitive dependence on initial conditions, and nonperiodicity. In their seminal paper [1], Ott, Grebogi, and Yorke (OGY) have shown that it is possible to control a chaotic orbit to a desired fixed or periodic one using a feedback technique with small perturbations on accessible parameters of a system. Since the seminal work by OGY, chaos control methods have been further developed and applied to various experimental systems [2].

In contrast to the OGY method which stabilizes unstable periodic orbits, the constant feedback (CF) method [3] and the proportional pulses (PP) method [4] convert chaotic orbits to stable periodic ones by an additional parameter, hence they do not require any *a priori* knowledge of the systems. The CF method has been used for a model of extinction and widely applied [5–8]. Recently, Melby *et al.* presented the self-adjusting logistic map (SALM) which also accomplishes stabilization of chaos by the use of the low-pass filtered feedback from the system variable with the accessible parameter [9]. However, these methods require the accessible parameter of the object system in SALM and the external adjustment of the additional parameter in the CF and PP methods for controlling chaos.

This paper describes a method that stabilizes chaotic orbits into periodic ones using the flexibility of chaotic dynamics without specifically modeling the system. The advantages of the method, compared with the previous feedback methods [3,4,9], are as follows. The proposed method utilizes the essence of the CF method, that is, any *a priori* knowledge of the systems and even accessible parameters are not necessary, and therefore can be more widely applied to chaotic systems than the method in SALM. Furthermore, the proposed method works without any additional technique such as the low-pass filter in SALM, and only needs a simple feedback with observations and comparisons of the values of the system variable. On the other hand, unlike the CF and PP methods, in the proposed method the additional parameter is no more constant with time but adjusts its value by a feedback from the variable. The time-variant parameter is changing more slowly than the state variable in the feedback adjustment. This slow dynamics is determined according to a largest value observed over a fixed time interval of time series of the internal state variable.

In the CF method, the object system is described as follows:

$$x_{n+1} = f(x_n, a) + k, \quad n = 0, 1, \ldots, N, \tag{1}$$

where $f(\cdots, a)$ is a one-dimensional unimodal function with a parameter a. x_n is the value of the state variable at time n, and k denotes an additional parameter for constant feedback. The chaotic orbit can be stabilized into a fixed or periodic one by adjusting the value of k externally. The stabilization has been proven theoretically [10].

In the proposed method, we suppose that the dynamical system (1) has the region with respect to the parameter k, where chaotic behaviors are observed with a great number of periodic windows. In general, this kind of bifurcation structure with periodic windows is ubiquitous for most smooth chaotic systems and observed in many experimental systems. Additionally, we assume that the chaotic behavior is generated through the upward curved part of the function f including a maximum with the ordinary period-doubling bifurcation. Explaining with examples, if $f(\cdots, a)$ is the logistic map with $f(x,a) = ax(1-x)$ or the exponential map with $f(x,a) = x\exp[a(1-x)]$ [11], periodic windows can be observed with respect to k [5,8]. These maps are upward curved functions.

For feedback adjustment to work in the system (1), we introduce dynamics to k with feedback from the state variable x_n. Thus, k changes its value with time, which is denoted by k_n. Furthermore, k_n is changing more slowly than x_n. In particular, k_n is updated every T steps of the time unit of x_n, where T is an integer parameter for a separation of time scales between x_n and k_n. This separation is necessary for the method to be self-contained without external control, that is, k_n governs the dynamics of x_n every T steps of time series similar to a control parameter, while the time series of x_n determine the dynamics of k_n. We consider the following system with the two-dimensional map:

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$$x_{n+1} = f(x_n,a) + k_n, \quad n = 0,1, \ldots ,N, \tag{2}$$

$$k_{n+1} = g_n(x_n,k_n), \tag{3}$$

where g_n is a feedback function depending on x_n, k_n, and n.

We introduce the following procedure to stabilize chaotic dynamics into periodic one in periodic windows. We fix the values of the parameters a and T, where a is determined to generate a chaotic behavior. We set the initial conditions for k as $k_0=0$ and for x as an arbitrary x_0. First, in order to determine the feedback function g_n, we estimate the upper bound of the chaotic attractor of the system (2) without taking into account k_n. We iterate f for $T'(\gg T)$ steps with the initial condition x_0, where T' is an integer parameter for the estimation of the upper bound. The largest value of the resulting time series $\{x_j\}_{1\le j\le T'}$ is denoted by x_e. Although x_e is usually less than the real upper bound x_m of the chaotic attractor, T' is set as large as possible such that $|x_m-x_e|$ is small enough, that is, the maximum value x_m is well approximated by x_e. We find a power law relation $\langle|x_m-x_e|\rangle \sim T'^{-\alpha}$, $\alpha\simeq 2$ for the logistic and exponential map, where $\langle\cdots\rangle$ is an average over 100 trials.

Next, we begin to iterate the map (2) with the same initial conditions x_0 and k_0 again. As k_n is changing every T steps according to Eq. (3), the feedback function g_n is defined as follows:

$$g_n(x_n,k_n) = \begin{cases} \hat{x}_i - x_e & \text{if } n = iT,\ i = 1,2,3, \ldots , \\ k_n & \text{if } n \neq iT, \end{cases} \tag{4}$$

$$\hat{x}_i = \max\{x_j\}_{(i-1)T<j\le iT}. \tag{5}$$

The schematic illustration of the method is shown in Fig. 1.

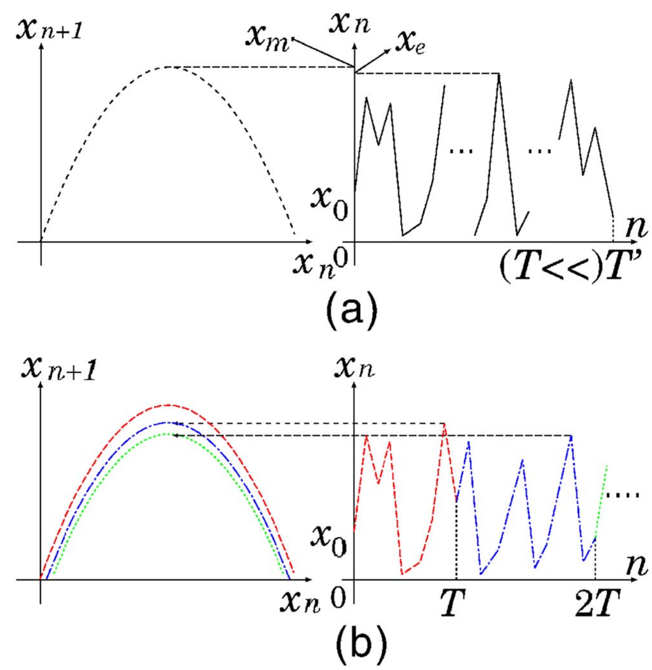

FIG. 1. (Color online) Schematic illustration of the method. (a) The estimation process of the maximum of the function f in Eq. (2). (b) The mechanism of the feedback function g_n. The first T steps of time series (right) is generated by the map described by the red (dashed) curve (left). Then, the map for the next T steps, which is denoted by the blue (dashed-dotted) curve (left), is determined with the largest value of the first T steps, such that the largest value corresponds to the maximum of the blue (dash-dotted) map. The same procedure is repeated every T steps.

The mechanism of the behavior of k_n in a periodic window as shown in Fig. 2(a), is elucidated schematically in Fig. 2(b). When $n=iT$, the feedback function g_n determines k_{n+1} which is the k coordinate of the cross point between the horizontal line through $\hat{x}_i$ and the locus of the maximum of $f(x)+k$ in the (k,x) plane. The locus is denoted by $x=h(k)$ and we call the "maximum line." In practice, using the estimated maximum value x_e, we can approximate the maximum line as follows:

$$h(k) \approx k + x_e. \tag{6}$$

Note that we can make this approximation for any function f as well, because we use the additive parameter k in Eq. (1). This mechanism makes it possible to apply our method to a broad class of smooth chaotic maps. The approximated maximum line is less than the real maximum line. Therefore, cross points exist between the approximated maximum line and the largest curves that are composed of the set of the largest values of periodic solutions, as shown in Fig. 2(b). The cross points correspond to the fixed points of the k_n dynamics, and are either stable or unstable. If the derivative of the largest curves at a cross point is less than 1, then the point is stable. The critical value 1 is the slope of the line described by Eq. (6). Otherwise, the point is unstable. Since there is no cross point in a chaotic region except periodic windows, the attraction of k_n to fixed points is possible only in periodic windows, that is, the orbit of k_n can be attracted by one of the cross points, as shown in Fig. 2(b).

Let us consider the variation of k_n. The values of k_{iT}, $i=0,1,2,\ldots$, are almost always decreasing with increasing i. If T' is large enough, i.e., if $|x_m-x_e|$ is small enough, $\hat{x}_i$ is almost always less than the approximated maximum ($k_{iT}+x_e$). Thus, from Eq. (4), $k_{iT}>k_{iT+1}(=k_{(i+1)T})$ in almost all i. However, some chaotic systems described by Eq. (1) may have a lower limit k_c such that whenever $k<k_c(<0)$ the orbit of x_n escapes from the chaotic attractor. Here we consider the system where such k_c exists. Under the assumption that the chaotic behavior is generated through the upward curved part of f including the maximum, the upper bound of the chaotic attractor is the maximum value $x_m(k)$ of the function $f(x,a)+k$, and the lower bound of the attractor, denoted by $x_l(k)$, is one time mapping of $x_m(k)$ by Eq. (1). In the case that there exists a fixed point $x_f(k)$ less than the critical point, $x_f(k)$ is the lower bound of the basin of the attractor. Thus, if $k=k_c$ then the lower bound of the attractor corresponds to that of the basin of the attractor, i.e., $x_l(k_c)=x_f(k_c)$. In addition, if $k<k_c$ then $x_l(k)<x_f(k)$, thus the orbit of x_n escapes from the chaotic attractor [see also the case of $k=k^2$ in Fig. 3(a)]. Consequently, if k_n becomes less than k_c, then k_n will diverge.

To prevent such a case, we keep k_n in a limited interval whose width is denoted by Δk. We modify the function g_n in Eq. (4) when $n=iT$, $i=1,2,3,\ldots$, such that k_n is kept in the interval $[k_0-\Delta k,k_0]$, as follows:

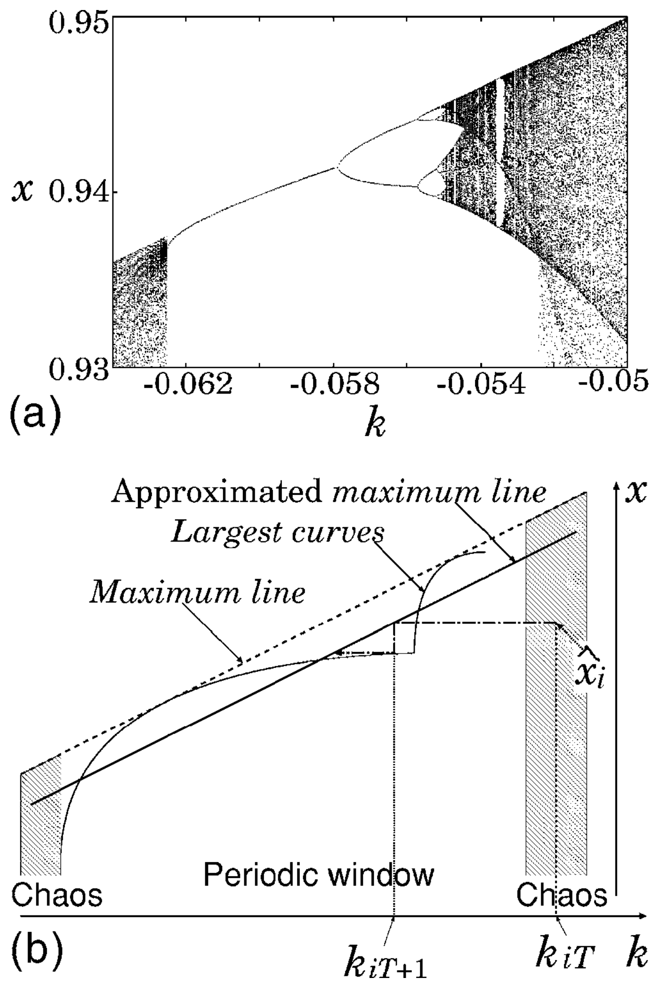

FIG. 2. (a) Upper part of a periodic window in the bifurcation diagram with respect to parameter k of Eq. (1) for the logistic map. (b) Schematic diagram illustrating convergence mechanism to a stable fixed point of k_n in a periodic window. The dashed-dotted lines show the orbit of k_n.

$$g_n(x_n,k_n) = k_0 - \tilde{k}_n \quad \text{if } n = iT, \tag{7}$$

$$\tilde{k}_n = |k_0 - (\hat{x}_i - x_e)| \quad (\text{mod } \Delta k). \tag{8}$$

In practice, we estimate an appropriate value of Δk according to the systems as follows. For the map where the orbit of x_n escapes from the basin of the attractor when $k < k_c$, if $k_c < k$ then the attractor still exists, and the following condition is satisfied: (I) $x_l(k) < F[x_l(k),k]$, where $F(x,k) = f(x,a)+k$. Otherwise, $x_l(k) > F[x_l(k),k]$, which corresponds to disappearance of the chaotic attractor [see also Fig. 3(a)]. Therefore, we determine the value of the lower bound of the interval $k_0-\Delta k$ as the value of k which satisfies the condition I and is as large as possible in the absolute value. For another class of maps without the escape from the basin of the attractor, the condition I is also valid for determining the appropriate value of $k_0-\Delta k$ [see also Fig. 3(b)]. Note that we cannot use the value x_m so that we calculate x_l with the estimated maximum value x_e.

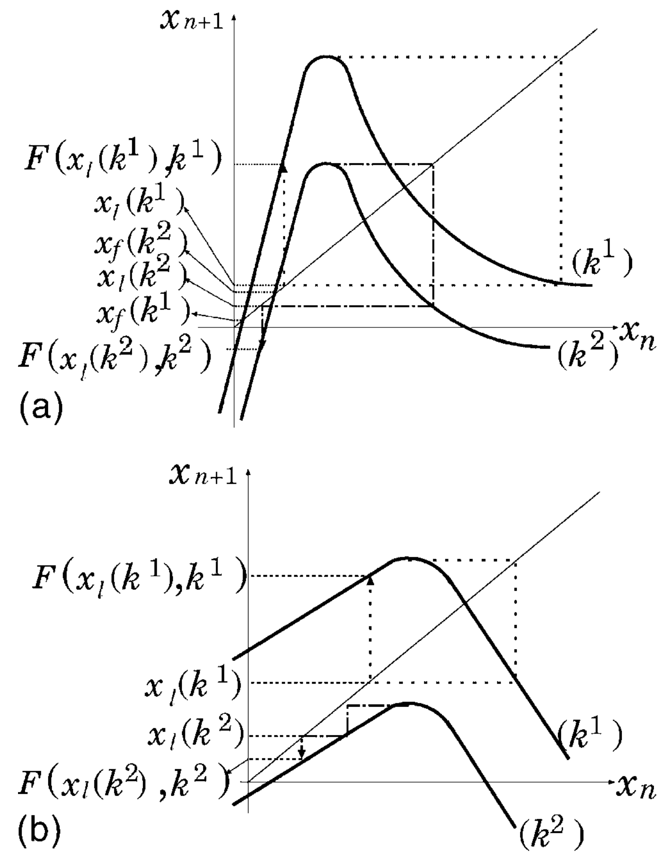

FIG. 3. Schematic illustration of the escape of x_n from the attractor and condition I in the (x_n,x_{n+1}) plane. (a) The condition I: $x_l < f[x_l(k),a]+k$, is satisfied for the map with $k^1 (>k_c)$, while not satisfied for the map with $k^2 (<k_c)$. For the map with k^2, the orbit indicated by dashed-dotted lines escapes from the basin of the attractor including the maximum. (b) In the case without the escape of x_n from the basin of the attractor, the appropriate value of the lower bound $k_0-\Delta k$ can be also determined between k^1 and k^2 by condition I.

This modification assures the convergence to a periodic window for a number of different initial values and for several systems in numerical simulations. Table I shows convergence rates for 10 000 initial values of x_0. In the case of the logistic map and the exponential map, we chose initial values from the interval $[0,1]$. In the case of the chaotic neuron map [12] with $f(x,a) = 0.7x - 1/[1+\exp(-x/0.02)]+a$ as an example of bimodal mapping, we chose them from the interval $[-1,0]$. The parameter values and the initial setting are as follows: $a=4$ (logistic map), $a=2.8$ (exponential map),

TABLE I. Rates of convergence of k_n for 10 000 initial states of x_0 with an accuracy of 10^p. The Lyapunov exponent λ is calculated by the value of k_N.

	$p=-5$	$p=-6$	$p=-7$	$\lambda<0$
Logistic map	0.9663	0.9408	0.9379	0.9585
Exponential map	0.9727	0.963	0.9612	0.997
Chaotic neuron nap	0.9776	0.9763	0.9752	0.9902

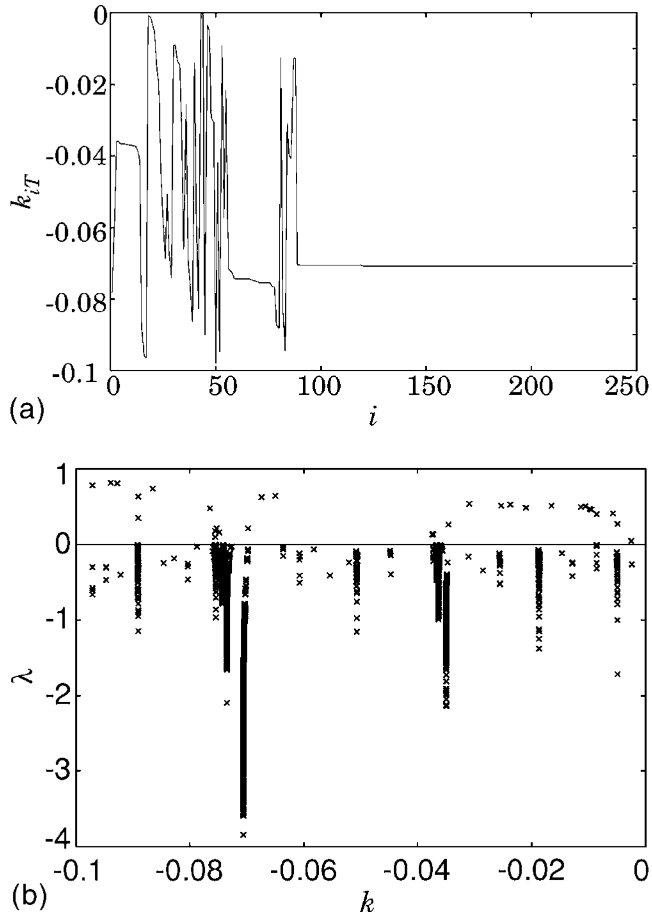

FIG. 4. (a) Time series of k_n (when $n=iT$) showing a self-adjusting process to a value at which a periodic state is generated. The time series data are shown till $i=250$. (b) Distribution of Lyapunov exponents with respect to the converged values of k_n in Table I in the case of the exponential map. The horizontal and the vertical axes show the original k and the Lyapunov exponent λ, respectively.

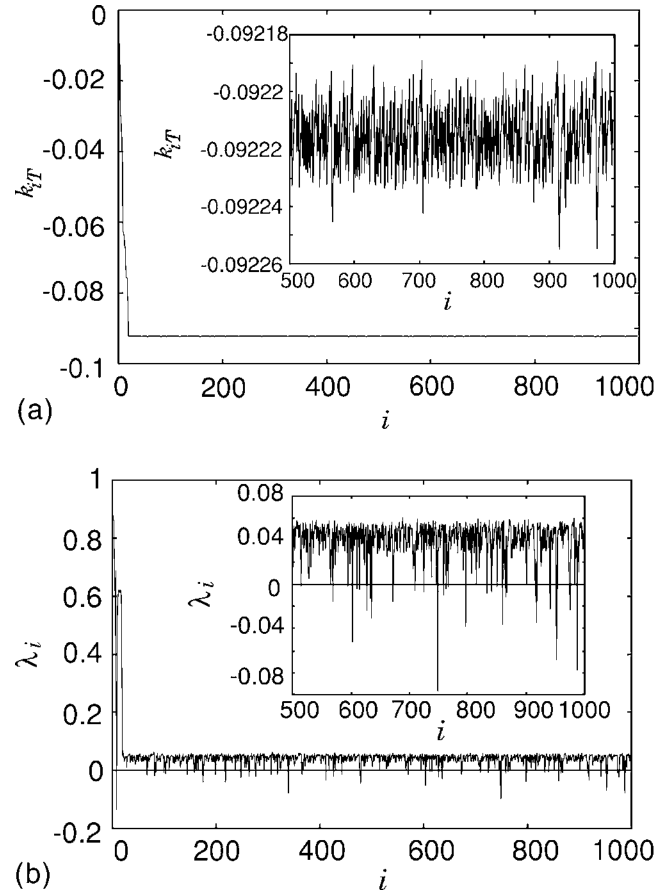

FIG. 5. (a) Time series of k_{iT} in one case of $\lambda>0$ in Table I with the inset of the enlargement in $[500,1000]$ of i. (b) The time series of the Lyapunov exponent λ_i corresponding to k_{iT} with the inset of the enlargement in the same interval.

$a=0.35$ (chaotic neuron map), $T=20$, $T'=1000$, and $N=20\,000$. The width of the interval Δk is 0.499, 0.0989, and 0.361 for the logistic, exponential, and chaotic neuron maps, respectively. In Table I, several values are chosen for the precision of the convergence p [13]. Furthermore, the convergence to periodic windows is confirmed by the Lyapunov exponent λ corresponding to the value of k_N. We can confirm that k_n have converged to periodic windows for almost all initial values x_0 as listed in Table I. Figure 4(a) shows the process of feedback adjustment as time series of k_{iT} for i in the case of the Exponential map. Since k_n can move almost everywhere in the domain $[k_0-\Delta k, k_0]$ because of the topological transitivity of chaos, k_n will go into a periodic window within a finite time. In fact, the convergence for most initial values as shown in Table I implies that k_n can be expected to go into one of periodic windows.

With respect to the determination of the strength k_n of the feedback, our method is more efficient than the CF method, because our method can *automatically* detect the locations of periodic windows, even narrow ones, as shown by negative values of λ in Fig. 4(b). On the other hand, the CF method needs an additional bifurcation analysis with respect to k for detecting periodic windows.

We can confirm numerically that k_n is almost converging to periodic windows for any initial values x_0. However, some of them have small positive Lyapunov exponents. Figure 5(a) shows the time series of k_{iT} for i with $\lambda>0$. Although k_{iT} is converging with an accuracy of about 10^{-5}, it is still fluctuating in a very narrow range as shown in the inset. Figure 5(b) shows the time series of the Lyapunov exponent λ_i which is calculated for each k_{iT}. We can observe that λ_i is also fluctuating around zero. However, the values of k_n are fluctuating around the value corresponding to a periodic window. The mechanism of the fluctuation is due to the approximation in Eq. (6).

We have shown numerically that the proposed method stabilizes a chaotic orbit into periodic one in a periodic window using the simple feedback adjustment. Since the assumptions in the proposed method are not so restricted, stabilization of chaos with the method might be expected to be achieved for a broad class of chaotic systems [14]. Furthermore, we also imply that our method provides adaptation to periodic states which exist around boundaries between regular states and chaotic ones, i.e., the edge of chaos. In fact, the results in Fig. 5 show that the value of k_n is fluctuating in a very narrow range over the edge of chaos that is represented as the zero Lyapunov exponent. The edge of chaos is considered to be an optimal condition for computation [15]. Although there are some previous methods exhibiting adaptation to the edge of chaos [9,16,17], the proposed method is simpler than such methods from the point of view that it dose not require any information on the systems and any additional techniques in Refs. [9,16,17].

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